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Subordination on ultrametric spaces

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Abstract. Using the formalism of continuous-time random walks (CTRW) we discuss particle diffusion on ultrametric spaces (UMS). For broad waiting time distributions $\psi(t) \sim t^{-\alpha-1}$ the temporal (α) and the energetic (γ) parameters combine multiplicatively (subordinate): i.e. $S(t)$, the mean number of UMS sites visited, follows a $S(t) \sim t^{\alpha\gamma}$ law. This mirrors our previous findings for CTRW on fractals.

1. Introduction

Recently, much interest has centred on ultrametric spaces UMS [1-6] as a means to model the energetic disorder found in amorphous media [7]. Thus the UMS complement the panoply of models for spatial randomness (geometric fractals) and for temporal disorder (continuous-time random walks—CTRW, multiple trapping—MT) and allow us to treat energetic randomness in systematic fashion. This paper deals with regularly multifurcating trees; one example is shown in figure 1. The point at the top multifurcates in b branches (here $b=3$) then every branch multifurcates again, and so on, until the baseline is reached. The points lying on this baseline form our UMS. Thus, in finite trees, the number N of points belonging to the UMS is given by $N = b^n$ where n is the number of branching levels. The distance between two levels is a measure for the energy required to reach one branch from the other one. Of special interest are trees where this distance is a constant, Δ, between all levels, and where the relaxation is thermally assisted.

In previous works [3-5] the connection between random walks on fractals and on UMS has been drawn: to the spectral dimension $\tilde{d}$ for fractals [8, 9] corresponds on regularly multifurcating UMS the quantity 2γ, with

$$\gamma = (kT/\Delta) \ln b \tag{1}$$

where T is the temperature, Δ is the regular energy spacing and b is the number of subbranches.

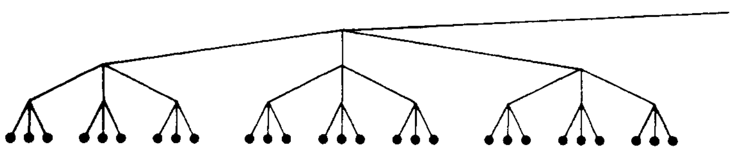

Figure 1. The regularly multifurcating UMS, $b=3$.

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CTRW on fractals shows subordination [10]: for waiting time distributions with long tails

$$\psi(t) \sim 1/t^{1+\alpha} \qquad t \text{ large} \tag{2}$$

one finds for the mean number of distinct sites visited $S(t) \sim t^{\alpha\tilde{d}/2}$ ($\alpha < 1$, $\tilde{d} < 2$), i.e. the exponents combine multiplicatively. Here we show that a similar situation occurs when one considers CTRW on UMS (and thus combines the temporal with the energetic facets of disorder). As a by-product of our analysis we obtain the exact closed-form generating function [11, 12] $P_0(z)$ for random walks on regularly multifurcating UMS. From this quantity the mean number S_n of distinct sites visited in n steps can be calculated exactly as we proceed to show. The central input for our analysis is $P_0(t)$, the probability of being at the origin of the walk at a later time t, a quantity which for regular UMS was determined exactly [2, 6].

The main concern of this paper is the CTRW aspect and the interplay between CTRW and RW in discrete time; thus we first give a summary of RW and CTRW properties essential for us.

2. Connection between RW in discrete time and CTRW

Let us recall the basic idea [11-13] of the connection between RW and CTRW. We exemplify it using $P_{0,n}$, the probability of being at the origin at the nth step. Let $\varphi_n(t)$ be the probability of having performed exactly n steps in time t. Then

$$P_0(t) = \sum_{n=0}^{\infty} \varphi_n(t) P_{0,n}. \tag{3}$$

The $\varphi_n(t)$ may now be expressed through $\psi(t)$, the waiting time distribution between steps. Setting $f(u) \equiv \mathscr{L}(f(t))$ where $\mathscr{L}$ is the Laplace transform, one has [12]:

$$\varphi_n(u) = (1-\psi(u))(\psi(u))^n/u. \tag{4}$$

Therefore:

$$P_0(u) = \frac{1-\psi(u)}{u} \sum_{n=0}^{\infty} (\psi(u))^n P_{0,n} \tag{5}$$

which, apart from the factor $(1-\psi(u))/u$ is nothing else but the generating function of the $P_{0,n}$

$$P(z) = \sum_{n=0}^{\infty} P_{0,n} z^n \tag{6}$$

evaluated at $z = \psi(u)$ [11]. Hence, one may switch from $P_{0,n}$ to $P(z)$ to $P_0(t)$ if only one of them is known. Furthermore, the same is true for S_n, $S(z)$ and $S(t)$. The following relation connects all quantities:

$$S(z) = 1/[(1-z)^2 P(z)]. \tag{7}$$

Equation (7) holds for regular lattices [11] (but not, in general, for fractals). In § 4 we derive equation (7) for regular UMS.

To close this section we mention that Poisson processes have exponentially distributed sojourn times $\psi(t) = \lambda \exp(-\lambda t)$, from which

$$\psi(u) = \lambda/(u+\lambda) \sim 1 - u/\lambda \tag{8}$$

follows, whereas long-time tails such as $\psi(t) \sim t^{-1-\alpha} (0 < \alpha < 1)$ lead to the following small-u expansion in the Laplace domain [12, 13]

$$\psi(u) \sim 1 - \Gamma(1-\alpha) u^{\alpha} / \alpha. \tag{9}$$

3. The generating function $P(z)$

As mentioned in the introduction, we treat uniformly multifurcating UMS. Furthermore, we restrict ourselves to the case of equidistant energy levels (say distant by Δ), and to temperature-assisted random walks. These assumptions are for computational convenience only, since in the general case the analysis may be carried out along the same lines. Now the transition rates ε_{ij} between sites i and j depend on the lowest energy barrier $m\Delta$ which separates the sites via

$$\varepsilon_{ij} = \tau^{-1} b^{-m} \, \mathrm{e}^{-m\Delta/kT} \equiv (R/b)^m / \tau \tag{10}$$

with $R = \exp(-\Delta/kT)$. The $P_i(t)$ fulfil the master equation

$$\frac{\mathrm{d}P_i(t)}{\mathrm{d}t} = \sum_{j=0}^{N} \varepsilon_{ij} P_j(t) \tag{11}$$

where $\varepsilon_{ii} = -\Sigma_{j \neq i} \varepsilon_{ij}$. For $P_i(0) = \delta_{i0}$ the solution $P_0(t)$ is [6]:

$$P_0(t) = b^{-n} + (b-1) \sum_{m=1}^{n} b^{-m} \exp(-\lambda_m t/\tau) \tag{12}$$

with

$$\lambda_m = (1 - R/b) \sum_{p=m}^{n-1} R^p + R^n. \tag{13}$$

For $b = 2$ and $\tau = 1/b$ the solution corresponds to that of [2]. In our further treatment, we will consider an infinite tree, $n \to \infty$. Equation (12) then takes the form of a Weierstrass series [7]:

$$P_0(t) = (b-1) \sum_{m=1}^{\infty} b^{-m} \exp[-R^m (t/\tau)(b-R)/(bR-R)]. \tag{14}$$

Note that $P_0(Rt) \sim bP_0(t)$, from which $P_0(t) \sim t^{-\gamma}$ with $\gamma = (\ln b)kT/\Delta$ (i.e. equation (1)) follows.

To obtain the generating function $P(z)$ of the walk we first need the Laplace transform of $P_0(t)$. From equation (12) for an infinite UMS one has:

$$P_0(u) = (b-1) \sum_{m=1}^{\infty} b^{-m} (u + \lambda_m/\tau)^{-1}. \tag{15}$$

Now, leaving a UMS site is an exponential process. Its rate is given from (10) by summing over UMS levels:

$$\lambda = \frac{1}{\tau} \sum_{m=1}^{\infty} (b^m - b^{m-1})(R/b)^m = R(b-1)/(b-bR)\tau \tag{16}$$

and is independent of the particular site. The corresponding waiting time distribution is given by equation (8). Setting $z \equiv \psi(u) = \lambda/(u+\lambda)$, i.e. $u = \lambda(1-z)/z$, we obtain from (5) and (15) the generating function $P(z)$

$$P(z) = (b-1) \sum_{m=1}^{\infty} b^{-m}(1-z+CzR^m)^{-1} \tag{17}$$

with $C = (b-R)/(bR-R)$. Expanding (17) as a series in z it follows:

$$P_{0,n} = (b-1) \sum_{m=1}^{\infty} b^{-m}(1-CR^m)^n$$

$$\simeq (b-1) \sum_{m=1}^{\infty} b^{-m} \exp(-nCR^m) \tag{18}$$

a special case of $P_{0,n} \simeq P_0(\lambda t)$ for Poisson processes. The knowledge of $P(z)$ enables us to determine $S(z)$ and to consider CTRW with broad waiting time distributions. This we proceed to show.

4. The mean number of distinct sites visited

We first show that relation (7), valid for regular lattices [11] also holds for UMS. The central point in this derivation is the relation

$$P_{i,n} = \sum_{m=1}^{n} P_{ii,n-m}F_{i,m} + \delta_{i0}\delta_{n0} \tag{19}$$

where $F_{i,m}$ is the probability to reach site i for the first time in the mth step. Equation (19) states (besides the obvious initial condition) that in order to be in the nth step at i one has to arrive there either at the nth step or earlier, at m, followed by a return to i in $n-m$ steps (whose probability is $P_{ii,n-m}$). Equation (19) holds since the random walk is a homogeneous Markov process, invariant with respect to time-translation, and since event spaces corresponding to different first-time arrivals are disjoint sets. Due to translational invariance, for Bravais lattices one has $P_{ii,n-m} = P_{i-i,n-m} = P_{0,n-m}$. The same relation holds also for regularly multifurcating UMS, since all sites are equivalent. (On the other hand, in general for fractals the relation $P_{ii,n-m} = P_{0,n-m}$ is only approximate.) Hence also for UMS:

$$P_{i,n} = \sum_{m=1}^{n} P_{0,n-m}F_{i,m} + \delta_{i0}\delta_{n0}. \tag{20}$$

The course to equation (7) is now straightforward [11] and is left to the appendix. The final result is (A8) of the appendix, i.e. equation (7):

$$(1-z)S(z) = [(1-z)P(z)]^{-1}. \tag{21}$$

By expanding $P(z)$ and $S(z)$ in series in z and by comparison term by term one can (using equation (17)) determine the S_n to any desired accuracy [14]. Here we proceed analytically, and determine from equations (14)-(17) and (21) the qualitative behaviour of $S(u)$ and $S(t)$. Consider first the Poisson process, for which we already know (equation (14)) that $P_0(t) \sim t^{-\gamma}$. Laplace transformation gives as a leading term:

$$P_0(u) \sim \begin{cases} u^{\gamma-1} & \text{for} \quad \gamma < 1 \\ \text{constant} & \text{for} \quad \gamma > 1 \end{cases} \tag{22}$$

as may be inferred either directly or from (15). For the latter one notices first from (13) that $\lambda_m \sim R^m$. Equation (15) then takes the form $\Sigma_{m=1}^{\infty} b^{-m}/(u+R^m)$, so that for $u=0$ the series converges if and only if $bR>1$, i.e. $\gamma>1$. Furthermore, for $bR<1$ one has for the integral J:

$$J \equiv \int_0^{\infty} \mathrm{d}y\, b^{-y}/(u+R^y) = \frac{1}{|\ln R|}\int_0^{\infty} \mathrm{d}x\, \mathrm{e}^{-\gamma x}/(u+\mathrm{e}^{-x}) \sim u^{\gamma-1} \tag{23}$$

as can be established using equation (3.311.9) of [15]. The sum in (15) and J have the same small-u behaviour, so that (22) follows.

For the Poisson process $z=\lambda/(u+\lambda)\sim 1-u/\lambda$, so that $1-z\sim u/\lambda$. Equation (21) implies, together with (22):

$$S(u) \sim \begin{cases} u^{-\gamma-1} & \text{for} \quad \gamma<1 \\ u^{-2} & \text{for} \quad \gamma>1 \end{cases} \tag{24}$$

from which in the time domain follows

$$S(t) \sim \begin{cases} t^{\gamma} & \text{for} \quad \gamma<1 \\ t & \text{for} \quad \gamma>1. \end{cases} \tag{25}$$

Since the random walk has a Poisson distribution of waiting times, $n\sim\lambda t$, and hence $S_n\sim n^{\gamma}$ for $\gamma<1$ and $S_n\sim n$ for $\gamma>1$. This agrees with the previous analyses for random walks on UMS [4, 5]. One may note that for $\gamma<1$ the relation

$$S(t) \sim 1/P_0(t) \tag{26}$$

also holds, which is the hallmark of *compact* exploration [4, 5, 8, 10].

5. CTRW with broad waiting-time distributions

In this section we present the behaviour of CTRW on UMS where $\psi(t)\sim t^{-1-\alpha}(0<\alpha<1)$, so that $1-\psi(u)\sim u^{\alpha}$ (equation (9)).

We start by considering the probability of being at the origin, $P_0(t)$. Use of (5) and (17) leads to:

$$\begin{aligned} P_0(u) &= \frac{1-\psi(u)}{u}(b-1)\sum_{m=1}^{\infty} b^{-m}\frac{1}{1-\psi(u)+\psi(u)CR^m} \\ &\sim u^{\alpha-1}\sum_{m=1}^{\infty} b^{-m}\frac{1}{u^{\alpha}+CR^m}. \end{aligned} \tag{27}$$

An argument similar to the one advanced in (23) shows that

$$P_0(u) \sim \begin{cases} u^{\alpha-1}u^{\alpha(\gamma-1)} = u^{\alpha\gamma-1} & \text{for} \quad \gamma<1 \\ u^{\alpha-1} & \text{for} \quad \gamma>1 \end{cases} \tag{28}$$

from which it follows

$$P_0(t) \sim \begin{cases} t^{-\alpha\gamma} & \text{for} \quad \gamma<1 \\ t^{-\alpha} & \text{for} \quad \gamma>1. \end{cases} \tag{29}$$

For $\alpha<1$ and $\gamma<1$ the two coefficients combine multiplicatively in $P_0(t)$, i.e. the two processes *subordinate* [10]. Remarkable is also the fact that for $\gamma>1$ the long-time behaviour is dictated by the temporal disorder and that the energetic disorder is no longer important. A similar finding holds for CTRW on fractals with $\tilde{d} > 2$ where for $\alpha < 1$ the spectral dimension becomes an irrelevant parameter for the decay law at long times [10].

Let us now turn to the mean number of distinct sites visited. Similar to (5) one has for $S(u)$ under CTRW conditions:

$$S(u) = \frac{1 - \psi(u)}{u} \sum_{n=0}^{\infty} (\psi(u))^n S_n. \tag{30}$$

The S_n are as given after (25), $S_n \sim n^\beta$, with $\beta = \min(1, \gamma)$. Now [10]

$$\sum_{n=0}^{\infty} n^\beta z^n \simeq \int_0^\infty x\, e^{x \ln z}\, dx = \Gamma(\beta + 1)(-\ln z)^{-\beta - 1} \tag{31}$$

so that, for $1 - \psi(u) \sim u^\alpha$

$$S(u) \sim u^{\alpha - 1} u^{-\alpha(\beta + 1)} = u^{-\alpha\beta - 1} \tag{32}$$

i.e. $S(u) \sim u^{-\alpha\gamma - 1}$ for $\gamma < 1$ and $S(u) \sim u^{-\alpha - 1}$ for $\gamma > 1$. Hence

$$S(t) \sim \begin{cases} t^{\alpha\gamma} & \text{for} \quad \gamma < 1 \\ t^{\alpha} & \text{for} \quad \gamma > 1. \end{cases} \tag{33}$$

Again, $S(t)$ shows subordination for $\gamma < 1$. Furthermore relation (26), $S(t) \sim 1/P_0(t)$ is now obeyed for *all* γ (i.e. for all temperatures).

In summary, we have used $P(z)$, the generating function for random walks on regularly multifurcating UMS (equation (7)) to study the influence of broad distributions of waiting times. At low temperatures both the probability of being at the origin, $P_0(t)$, as well as the mean number of distinct sites visited, $S(t)$, show subordination. Findings on UMS parallel those for fractals; moreover some relations, such as (7) hold exactly for regular UMS.

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Appendix. The relationship between $P(z)$ and $S(z)$

The starting point for the relationship between $P(z)$ and $S(z)$ is (20):

$$P_{i,n} = \sum_{m=1}^{n} P_{0,n-m} F_{i,m} + \delta_{i0}\delta_{n0}. \tag{A1}$$

Now we define the increment Δ_m in newly visited sites at the mth step as being:

$$\Delta_m = \sum_{i \neq 0} F_{i,m} \tag{A2}$$

with $\Delta_0 = 1$. Then S_n, the mean number of distinct sites visited in n steps, is:

$$S_n = \sum_{m=0}^{n} \Delta_m. \tag{A3}$$

Equations (A1) and (A2) give by summing over $i \neq 0$:

$$\sum_{m=0}^{n} P_{0,n-m}\Delta_m = 1 \tag{A4}$$

where the requirement of conservation of probability was used.

Now we can switch over to generating functions. For that we multiply both sides by z^n and sum over n, so that equation (A4) becomes:

$$P(z)\Delta(z) = (1-z)^{-1} \tag{A5}$$

where

$$\Delta(z) = \sum_{m=0}^{\infty} \Delta_m z^m. \tag{A6}$$

Similarly, from equation (A3)

$$(1-z)S(z) = \Delta(z). \tag{A7}$$

Thus the last equation together with equation (A5) yield the sought relation

$$(1-z)S(z) = [(1-z)P(z)]^{-1} \tag{A8}$$

which is written here in a symmetric way.

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